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A

PRACTICAL METHOD

OF

COMPUTING THE LATITUDE AT SEA,

FROM

TWO OBSERVED ALTITUDES AND THE INTERVAL OF TIME BETWEEN.

BY THE

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A

PRACTICAL METHOD

OF

COMPUTING THE LATITUDE.

THE solution of this Problem by a convenient practical method has long been considered of great importance in Nautical Astronomy. This is shown by the many solutions that have been given.

The direct solution was at first considered as quite inapplicable to the uses of Seamen, and the indirect method of Douwes was the only one used by them till within a few years.

That Seamen acquiesced in and practised a solution so imperfect and limited, clearly shows the frequent necessity that occurs at Sea of having recourse to two Altitudes of the Sun and the interval of Time between, for finding the Latitude.

A peculiar set of Tables were constructed for this method by Douwes, which were afterwards very considerably extended by Admiral Campbell, and first published in the Nautical Almanac for 1772, and have been inserted in the subsequent editions of the "Requisite Tables." The necessary limitation of Douwes's method led me, in the year 1791,* to investigate an improvement by which that inconvenience might be avoided.

In the method of Douwes the Latitude is computed from using the Latitude by account in conjunction with the observed Altitudes, interval of Time and declination.

It is essential to his method that the Latitude thus computed should be much nearer to the true Latitude than the Latitude by account, so that by repeating the operations the Latitude may be obtained sufficiently near.

My method, for which I constructed two Tables, was published in the Nautical Almanacs for 1796, &c. By this method the true Latitude was obtained in most cases from the first result of Douwes, even when the computed Latitude was further from the truth than the Latitude by account.

The method, although as exact for nautical purposes as could be desired, is embarrassed by a necessary distinction of cases, by the use of decimal fractions not always well understood by Seamen, and by Two Tables with double arguments. Consequently it has not been found so generally convenient as was hoped.

The late Mr. Mendoza, in his valuable and extensive Tables, has given a direct method of Solution, which he conceived was sufficiently practical. This method requires extensive Tables, and Tables attended with inconveniencies. I shall not venture to give an opinion whether it be more practically convenient than my method above referred to; but it is certain that persons very conversant in Nautical Astronomy desire a method still more practically convenient.

* This method is noticed by the late Mr. Mendoza, in the Conn. des Tems. 1792. —I had given him a Copy of the Method and Tables in 1791.

I hope the following method will be found both simple and practically convenient, and be considered a useful Solution of this important Problem.

It is derived from the former Solution which I gave; but the practical rules now deduced contain no embarrassing distinction of cases, and require no Tables but the common Trigonometrical Tables to be found in the "Requisite Tables." Three places of figures besides the index only are necessary.

The method would have appeared shorter had I adapted my former Tables to it, but it would not really have been so, and it appears to me an object of great importance to avoid Tables with double arguments that require proportional parts, whenever it can be done.

In the first examples that I have given, I have supposed the Latitude computed by the method of Douwes by help of the Tables given in the "Requisite Tables;" but it may be doubted whether these Tables and the terms therein used may not tend to embarrass. Every thing performed by these Tables may, with scarcely any additional trouble, be obtained by the common Log. Tables of Sines, &c. and Tables ought not to be multiplied unless a decided advantage is thereby obtained.

That a choice may be made: I have given a rule and examples of the entire solution of the Problem by the common Tables of Natural and Logarithmic Sines, &c. It has been said, that in the same Solution both Natural and Logarithmic Sines ought not to be joined. I see no just cause for this objection. The object to be attained by the introduction of Logarithms is to avoid multiplication and division, and where Natural Sines are used in addition and subtraction only, there appears no cause for reducing them to Logarithmic quantities.

The following Rules for correcting the computed Latitude require:—

1. That the middle time do not exceed six hours from noon.*
2. That it be known, when the Declination and Latitude are nearly equal, which is the greater.

To apply this method with most advantage; it is recommended

1. That the middle Time do not exceed three Hours.
2. When the Latitude is greater than the Declination or of a contrary name, that both observations be made on the same side of Noon.
3. When the Latitude is less than the Declination and of the same name, that the observations be made on different sides of Noon.

When these circumstances are attended to, the Latitude and Declination may be within three or four degrees of each other, and the Latitude by account be inexact by a whole degree, and yet one operation will give the Latitude sufficiently exact.

If these circumstances be not attended to, that one operation may serve it will be necessary that the Declination and Latitude should differ by several degrees, and that the Latitude by account shall not err more than 20′ or 30′, otherwise it may be necessary to repeat the operation using the corrected Latitude as the Latitude by account.

* It may be useful in high Latitudes to apply this Problem when the middle Time is more than six hours from Noon. The necessary alteration in the Rule is afterwards given.

It is scarcely necessary to observe, that in general the interval between the observations should not be too small, so that the errors of observation may not too much affect the difference of Altitudes. Also that this and all other methods, even the direct one, become inconvenient and inexact when the Latitude and Declination are nearly equal.

The Rule for correcting the Latitude once computed is first given. Then examples are given both when the Latitude is first computed by the method as given in the "Requisite Tables," and when computed by another Rule here given, which it is conceived may be more convenient.

Rule for correcting the Latitude.

L, *P*, *T*, &c. signify Logarithms used in the computation.

Three places of figures in the Logarithms are sufficient.

The middle time, and half the time nearest noon, are to be reduced to Degrees and Minutes.

1.—*To find L.*

Add together the cotangent of the Latitude by Account and tan. of the Declination, call half the sum *P*.

(1) *Declination and Latitude of contrary names.*

Find *P* among the tangents, and take out the corresponding secant, which call + *L*.

(2) *Declination and Latitude of the same name, but Latitude greater than Declination.*

Find *P* among the sines, and take out the corresponding cosine, which call + *L*.

(3) *Declination and Latitude of the same name, but Latitude less than Declination.*

Find *P* among the secants, and take out the corresponding tangent, which call — *L*.

2.—*To find T.*

Add together the tan. middle time and cot. $\frac{1}{2}$ time nearest noon, call half the sum Q.

(1) *Observations on the same side of Noon.*

Look for Q among the secants, and take out the corresponding tangent; add this tang. the constant log. 0.150, and sine of $\frac{1}{2}$ time nearest noon together, taking 10 from the index, subtract this sum from 20.000, and mark the remainder – *T*.

(2) *Observations on different sides of Noon.*

Look for Q among the tangents, and take out the corresponding secant; add this secant, the constant log. 0.150, and sine of $\frac{1}{2}$ the time nearest noon together, taking 10 from the index; subtract this sum from 20.000, and mark the remainder + *T*.

3.—*To find C.*

Add *L* and *T* together, call the sum *S*, after having taken 10 from the index. *S* is to be marked + when *L* and *T* have like signs; and – when unlike.

(1) If *S* be marked –, look for it in the tangents, and take out the secant corresponding, which mark – *C*.

(2) If S be marked +, and be not less than 10.000, look for it in the secants, and take out the tangent corresponding, which mark + C.

(3) If S be marked +, and be less than 10.000, look for it in the cosines, and take out the sine corresponding, which mark − C.

4.—*To find the corrected Latitude.*

Find the Log. of the difference in Minutes between the computed Latitude, and that by account, which call D, and mark it + when the computed Latitude is the greater, and − when the less.

Double C, (rejecting the tens in the index) and take the arithmetical complement, which is to be added to D. The sum, rejecting ten in the index if it occur, is the Logarithm of the correction of the computed Latitude. Which correction is to be added to the computed Latitude when C and D have the same signs, otherwise it is to be subtracted.

N.B. When S is marked +, and is nearly = 10.000, this Rule cannot be conveniently practised.

This case cannot occur when the Observations are on the same side of Noon, and the Latitude and Declination are of different names, or the Latitude of the same name and greater than the Declination.

It cannot occur when the Observations are on different sides of Noon, and the Latitude less than the Declination, and of the same name.

EXAMPLE I.

Alt. 50°. 1′ A.M. } Interval 3h. 30′ { Lat. by Account, 51°. 30′ N.
41 33 P.M. } { ☉'s Declination 14 0 N.

By the Method in the Requisite Tables.

Computed Latitude .... = 52°. 4′
Middle Time......... 0h. 44′. 28″ ...... = 11 7
Half Time nearest Noon 0 30 16 ...... = 7 34

cot. 51°. 30′...... 9.901
tan. 14 0 ...... 9.397

2) 19.298

sin. 9.649 P.

cos. 9.952 + L.
10.534 + T.

sec. 10.486 + S.
tan. 10.463 + C.
2

0.926
A.C. 9.074

tan. 11°. 7′...... 9.293
cot. 7 34 ......10.877

2) 20.170

tan. 10.085 Q.

sec. 10.197
0.150
sin. 7°. 34′ 9.119

9.466
10.534 + T.

Lat. by acct. 51°. 30′
Computed .. 52 4

34 log. 1.531 + D.
9.074

Add corr. Latitude 4′ log. 0.605
Comp. Lat....... 52 4

Corrected Latitude 52 8

EXAMPLE II.

Alt. 5°. 36′ A.M. 45 6 A.M.	Interval 3h. 0′	Latitude by Account 27° 0′ N. ⊙'s Declination.... 12 0 N.

By the Method in the Requisite Tables.

Computed Latitude ....= 30°. 2′
Middle Time .... 4h. 25′. 12″ .... = 66 18
Half Time nearest Noon .. 1 27 36 .... = 21 54

cot. 27°. 0′......10.293
tan. 12 0...... 9.327

2) 19.620

sin. 9.810 *P*.

cos. 9.883 + *L*.
9.944 — *T*.

tan. 9.827 — *S*.
sec. 10.080 — *C*.
2

0.160
A. C. 9.840

tan. 66°. 18′......10.357
cot. 21 54......10.396

2) 20.753

sec. 10.376 *Q*.

tan. 10.334
0.150
sin. 21°. 54′...... 9.572

10.056
9.944 — *T*.

Comp. Lat......30°. 2′
Lat. by Acct...27 0

3 2 = 182′ log. 2.260 + *D*.
9.840

2.100

126′ log.

Corr. Latitude subtract..... 2°. 6′
Comp. Latitude........... 30 2

Corrected Lat. 27 56

The middle Time here is without the limit recommended, and as the computed Latitude and Latitude by account differ by three degrees, it will be more exact to use 27°. 56′ the corrected Latitude as the Latitude by account: then by the Method in the Requisite Tables—

The new computed Latitude = 28°. 6′
27 56

10 log. 1.000 + *D*.
A.C. 9.840 before found.

Subtract...... 7′ log. 0.840
28°. 6

27 59 Latitude twice corrected, which s exact.

It will seldom occur in practice, that a second correction will be requisite. But using the corrected Latitude as the Latitude by account, and finding by the operation in the Requisite Tables the computed Latitude, a satisfactory proof will be obtained, and also the exact Time may be had.

It is an important circumstance that, by this extension of the Method of finding the Latitude from two Altitudes and the Time between, a convenient practical Method of obtaining the Time, with sufficient exactness, is thereby gained. In the former limitation of the problem this could not be had, the errors of observation having a considerable influence on the Time. But here, as both observations may be considerably distant from Noon, the Time will not be so much affected by the Errors of Observation.

RULE

For finding the Latitude from two Observations,

BY THE COMMON LOGARITHMIC TABLES.

---

1. REDUCE half the interval between the observations to degrees and minutes.

2. Add together the Log. secant of the Declination, secant of Latitude, and 4.69897, (rejecting 20 from the index); call the sum *A*.

3. Add together *A*, the Logarithm of the difference of the natural Sines of the Altitudes (the natural Sines found in the "Requisite Tables" to be considered as whole numbers, the radius being 100000,) and the arithmetical comp. of the Log. Sine of half the interval, the sum is the Log. Sine of the middle Time in degrees and minutes.

4. The difference of this middle Time and of half the interval is the Time nearest Noon.

5. Double the Logarithmic Sine of half the Time nearest Noon, rejecting 10 from the index; from which double subtract *A*, the remainder is the Logarithm of a number to be added to the natural Sine of the greater, to find the Sine of the meridian Altitude.

6. From this meridian Altitude find the Latitude, which may be called the computed Latitude, and may be corrected by the Rule that has been given.

EXAMPLE I.

Alt. 76°. 6′ A.M. } Interval 6h. 20′ { Latitude by account 9° N.
Alt. 8 3 P.M. } { Declination . . . . . 20 N.

Half interval = 3h. 10′ = 47°. 30′

sec. 20° 10.02701
sec. 9 ......10.00538
4.69897

A. 4.73136
83068 log. 4.91943
Half interval, sin. 47°. 30′ A.C. 0.13257

Mid Time, sin. 37 22 9.78316

Time nearest Noon 10 8

Half 5 4 sin. 8.94603
2

7.89206
4.73136

1448 log. 3.16070

Comp. Latitude........10°. 8′
7

Latitude corrected......10 1
The true Latitude is....10 0

76°. 6′ Nat. sin. 97072
8 3 Nat. sin. 14004

Diff. 83068

cot. 9°. 0′ 10.800
tan. 20 0 9.561

2) 20.361

sec. 10.180 P.

tan. 10.056 — L.
10.413 + T.

tan. 10.469 — S.
sec. 10.493 — C.
2

0.986

A. C. 9.014
68 log. 1.833 + D.

Subtract.. 7 log. 0.847

97072
1448

80°. 8′ Nat. sin. 98520

80°. 8′

9 52 Z. D.
20 0 Declin.

Computed Latitude 10 8
By account 9 0

1 8 = 68′

tan. 37°. 22′ 9.883
cot. 5 4 11.052

2) 20.935

tan. 10.467 Q.

sec. 10.491
0.150
8.946

9.587
10.413 + T.

EXAMPLE II.

Alt. 21°. 26′ A.M. }
Alt. 60 56 A.M. } Interval 3^h. 0′ { Latitude by account 19° N.
{ Declination....... 1 N.

Half interval = 1^h. 30′ = 22°. 30′

	sec. 19°		10.02433
	sec. 1		10.00007
			4.69897
		A.	4.72337
	50864	log.	4.70641
	22°. 30′	sin. A.C.	0.41716
Mid. Time....	44 40	sin.	9.84694
Time nearest N.	22 10		
	11 5		
		sin.	9.28384
			2
			8.56768
			4.72337
	6987	log.	3.84431

60°. 56′	Nat. sin.	87406
21 26	Nat. sin.	36542
		50864
		70°.43′
Z. dist..............		19 17
Declination		1 0
Comp. Latitude......		20 17
By Account.........		19 0
		1 17 = 77′
Comp. Latitude....		20°. 17′
Subtract		18
Corrected Latitude..		19 59
The true Latitude is		20 0

	87406
	6987
Nat. sin.	94393

cot. 19°	10.463	
tan. 1	8.242	
2)	18.705	
sin.	9.352	P.
cos.	9.999	+ L.
	10.264	— T.
tan.	10.263	— S.
sec.	10.320	— C.
	2	
	0.640	
A. C.	9.360	
77′ log.	1.886	+ D.
Subt. 18′ log.	1.246	

tan. 44°. 40′	9.995	
cot. 11 5	10.708	
2)	20.703	
sec.	10.350	Q.
tan.	10.302	
	0.150	
	9.284	
	9.736	
	10.264	—T.

The two following Examples are without the limits recommended, and in such cases one or two repetitions of the Operation may be necessary. But it is conceived that the additional trouble will, in general, be fully compensated by the results.

EXAMPLE III.

Alt. 70°. 1′ A.M.
Alt. 35 21 A.M.
} Interval 2^h. 20′ {
Latitude by Account 6°. 30′ N.
Declination········ 5 30 N.

Half interval = 1^h. 10′ = 17°. 30′

	sec. 6°. 30′		10.00230
	sec. 5 30		10.00200
			4.69897
		A.	4.70377
	36122	log.	4.55777
	sin. 17°. 30′	A. C.	0.52186
Mid. Time....sin.	37°. 24′		9.78340
Time nearest Noon	19 54		
Half	9 57		
		sin.	9.23751
			2
			8.47502
			4.70377
5905		log.	3.77125
93979			
99884	sin. 87°. 15′		
	2 45 Z. D.		
	5 30 Declination.		
	8 15 Comp. Latitude.		
	6 30 Latitude by account.		
	1 45 = 105′		

70°. 1′	Nat. sin.	93979	
35 21	Nat. sin.	57857	
		36122	
cot. 6°. 30′		10.943	
tan. 5 30		8.984	
	2)	19.927	
	sin.	9.963	*P.*
	cos.	9.597	+ *L.*
		10.350	— *T.*
	tan.	9.947	— *S.*
	sec.	10.126	— *C.*
		2	
		0.252	
		9.748	
105 log.		2.021	+ *D.*
Subtract 59′	log.	1.769	
8°. 15			
7 16 Latitude.			

tan. 37° 24′		9.883	
cot. 9 57		10.756	
	2)	20.639	
	sec.	10.319	*Q.*
	tan.	10.262	
		0.150	
sin. 9°. 57′		9.238	
		9.650	
		10.350	— *T.*

Again, taking 7°. 16′ as the Latitude by Account—

sec. 7°. 16′		10.00350
sec. 5 30		10.00200
		4.69897
	A.	4.70447
		4.55777
sin. 17°. 30′	A. C.	0.52186
sin. 37 28		9.78410
19 58		
Half 9 59	sin.	9.23895
		2
		8.47790
	A.	4.70447
5935		3.77343
93979		
99914	sin. 87°. 37′	

Z. Distance....	2 23
Declination....	5 30
Lat. 2 comp...	7 53
	7 16
	37

cot. 7°. 16′	10.894	
tan. 5 30	8.984	
2)	19.878	
sin.	9.939	*P.*
cos.	9.694	+ *L.*
	10.349	— *T.*
tan.	10.043	— *S.*
sec.	10.173	— *C.*
	2	
	0.346	
A. C.	9.654	
37′ log.	1.568	+ *D.*
Subt. 17′ log.	1.222	

7°. 53
7 36 = Latitude sufficiently exact: for if with this Latitude we recompute we shall find
Latitude computed .... 7°. 40′
And Latitude corrected.......... 7 39

tan. 37°. 28′	9.884	
cot. 9 59	10.754	
2)	20.638	
sec.	10.319	*Q.*
tan.	10.262	
	0.150	
sin. 9°. 59′	9.239	
	9.651	
	10.349	—*T.*

EXAMPLE IV.

Latitude by Account 22°. 30′ N.
Declination........ 19 0 N.

Altitude 80°. 10′ A.M. } Interval 5h. 20′
Altitude 24 57 P.M. }

Half interval = 2h. 40′ = 40°

Latitude once computed by Rule .... = 22°. 22′
Latitude corrected 22 4

With Latitude by Account.......22°. 4′
Latitude 2 computed 22 3
Latitude 2 corrected 22 0

In high Latitudes it may be convenient to apply this Method when the two Observations are between Six o'Clock and Midnight and on the same side of Midnight, and therefore the middle Time greater than Six Hours.

In this case the middle Time found, according to the Rule, is the middle Time from Midnight, and therefore is to be subtracted from 180° to find the middle Time from Noon.

In the Rule for correction of the Latitude, the only alteration is in the Rule for finding *T*, which is then as follows:—

Add together tan. middle Time from Midnight, and cot. half time nearest Noon, call half the sum *Q*.

Look for *Q* among the tangents, and take out the corresponding secant. Add this secant, the const. log. 0.150, and the sine of half time nearest Noon together (rejecting 10 from the index). This sum is then to be subtracted from 20, and the remainder marked + *T*.

EXAMPLE V.

Latitude by Account 81°. 10′ N.
Declination........ 23 4 N.

Altitude 16°. 12′ P.M.
Altitude 14 44 P.M.

By Watch 9h. 33′. 8″
By Watch 11 16 48

Interval 1 43 40

Half Interval = 51′. 50″ = 12°. 57′

sec. 81°. 10′........10.81372
sec. 23 4........10.03619
4.69897

A. 5.54888
2467 log. 3.39217
12°. 57′ sin A. C. 0.64956

16°. 12′ Nat. sin. 27899
14 44 Nat. sin. 25432

2467

Middle Time from Midnight 22°. 56′ sin. 9.59061
Middle Time from Noon... 157 4
Time nearest Noon........ 144 7
Half.................... 72 3

sin. 9.97833
2

9.95666
A. 5.54888
25573 log. 4.40778
27899

53472

M. Altitude..........32°. 20′

Z. D................57 40
Declination..........23 4

Computed Latitude.... 80 44
Latitude by Account..81 10

26

cot. 81°. 10′...... 9.191
tan. 23 4...... 9.629

2.) 18.820

sin. 9.410 *P.*

cos. 9.985 + *L.*
9.844 + *T.*

cos. 9.829 + *S.*
sin. 9.868 — *C.*
2

9.736

A. C. 0.264
26 log. 1.415 — *D.*

Add 48′ log. 1.679
Computed Latitude...... 80°. 44

Corrected Latitude...... 81 32 exact.

tan. 22°. 56′...... 9.626
cot. 72 3′...... 9.510

2.) 19.136

tan. 9.568 *Q.*

sec. 10.028
0.150
sin. 72°. 3′ 9.978

10.156
9.844 + *T.*

DEMONSTRATION.

BOTH OBSERVATIONS ON THE SAME SIDE OF NOON.

Let D = Declination.
l = Latitude.
t = Time nearest Noon.
m = Middle Time.
p = Interval between the observations.
A = Greater Altitude.
a = Less Altitude.
z = Zenith distance on meridian.

By spherical trigonometry,

$$\sin. A = \sin. D \sin. l + \cos. D \cos. l \cos. t \quad \cdots\cdots (1)$$
$$\sin. a = \sin. D \sin. l + \cos. D \cos. l \cos. (t + p) \cdots\cdots (2)$$

Hence,

$$\cos. t - \cos. (t + p) = \frac{\sin. A - \sin. a}{\cos. D \cos. l};$$

$$\text{or } \sin. \tfrac{1}{2} p \times \sin. m = \frac{\sin. A - \sin. a}{\cos. D \cos. l};$$

$$\text{or } \sin. m = \frac{\sin. A - \sin. a}{\cos. D \cos. l \sin. \frac{1}{2} p} \quad \cdots\cdots (3)$$

$$t = m - \tfrac{1}{2} p \cdots\cdots (4)$$

Now, $\cos. z = \cos. (l - D) = \cos. l \cos. D + \sin. l \sin. D$
and $\sin. D \sin. l = \sin. A - \cos. D \cos. l \cos. t.$
Hence $\cos. z = \cos. l \cos. D \text{ vsin. } t + \sin. A \quad \cdots\cdots (5)$
$= 2 \cos. l \cos. D \sin.^2 \tfrac{1}{2} t + \sin. A;$
computed lat. $(c) = z + D.$

From hence the Rule for finding the computed Latitude as given in the "Requisite Tables" will be easily deduced, as also the Rule that is given above as a substitute for that in the "Requisite Tables."

By last Equat.

$$d\, c = d\, z \cdots\cdots (6)$$

By Equat. (5) $d\, z = \dfrac{\cos. D.}{\sin. (l - D)} (\text{vsin. } t \sin\, l\, d\, l - \cos. l \sin. t\, d\, t).$

By Equat. (4) $d\, t = d\, m.$

By Equat. (3) $d \log. \sin. m. = - d \log. \cos. l,$
or $d\, m \cot. m = d\, l \tan. l,$
or $d\, m = d\, l \tan. l. \tan. m.$

Hence by substitution Equat. (6) becomes

$$d\, c = \frac{\cos. D\, d\, l \sin. l}{\sin. (l - D)} (\text{vsin. } t - \sin. t. \tan. m)$$

$$= \frac{d\, l}{1 - \tan. D \cot. l} (\text{vsin. } t - \sin. t \tan. m) = \frac{d\, l}{n};$$

or $d\, l : d\, c :: n : 1,$

$$\text{making } n = \frac{1 - \tan. D \cot. l}{\text{vsin. } t - \sin. t \tan m}.$$

What is true for the Fluxions will be nearly so for the Increments, and will be sufficiently true for Nautical purposes, even when the Increments are large.

Let r = the Latitude by account or reckoning.
t = the true Latitude.

Then $t-r : t-c :: n : 1$ nearly;
or $t-c : c-r :: 1 : n-1$,
or $t-c = \frac{c-r}{n-1}$.

This value of $t-c$ fails when $n=1$,
that is when $dc = dt$,
then $t-c = t-r$
and $c = r$.

Also when $n = 0$
for then $dc = \frac{dt}{0}$.

Now $n=0$ when $1 - \tan. D \cot. l = 0$,
$\tan.. D = \tan. l$
or $D = l$.

Also, when $\text{vsin.}\, t - \sin. t \tan. m$ is infinite; that is when $\tan. m$ is infinite, or $m = 90°$.

It is next required to facilitate the computation of $n-1$.

I.—For $1 - \tan. D \cot. l$.

(1) D and l of different Denominations, it becomes $1 + \tan. D \cot. l$, and therefore is always positive, and greater than unity.

Hence, let
$$1 + \tan. D \cot. l = 1 + \tan.^2 \alpha = \sec.^2 \alpha;$$
Then $\tan. D \cot. l = \tan.^2 \alpha$.

(2) D and l of the same Denomination, but l greater than D.
Then $1 - \tan. D \cot. l$ is always positive, but less than unity, therefore let $1 - \tan. D \cot. l = 1 - \sin.^2 \alpha = \cos.^2 \alpha$;
then $\tan. D \cot. l = \sin.^2 \alpha$.

(3) D and l of the same name, but D greater than l.
Then $1 - \tan. D \cot. l$ is always negative.
Let $1 - \tan. D \cot. l = (1 - \sec.^2 \alpha) = - \tan.^2 \alpha$;
$\tan. D \cot. l = \sec.^2 \alpha$.

II.—For $\text{vsin.}\, t - \sin. t \tan. m$

$$\text{vsin.}\, t - \sin. t \tan. m = \text{vsin.}\, t \left(1 - \frac{\sin. t}{\text{vsin.}\, t} \tan. m\right)$$
$$= -2 \sin.^2 \tfrac{1}{2} t \, (\cot. \tfrac{1}{2} t \tan. m - 1);$$
t is always less than m;
therefore, $\cot. \frac{1}{2} t \times \tan. m$ is always greater than unity.
Let $\cot. \frac{1}{2} t \times \tan. m = \sec.^2 \beta$
and then $\text{vsin.}\, t - \sin. t \tan. m = -2 \sin.^2 \frac{1}{2} t \tan.^2 \beta$.

Hence joining (1) and II.
$$n = \frac{\sec.^2 \alpha}{-2 \sin.^2 \frac{1}{2} t \tan.^2 \beta};$$
(2) and II.
$$n = \frac{\cos.^2 \alpha}{-2 \sin.^2 \frac{1}{2} t \tan.^2 \beta};$$
(3) and II.
$$n = \frac{-\tan.^2 \alpha}{-2 \sin.^2 \frac{1}{2} t \tan.^2 \beta}.$$

III.—To compute $n-1$.

(*a*) When n is negative
$-(n+1)$ is to be investigated, n being taken positively.
Let $n = \tan.^2\gamma$;
Then $1+n = \sec.^2\gamma$.

$$(1) \text{ and II.} \qquad \tan.\gamma = \sqrt{n} = \frac{\sec.\alpha}{\sqrt{2.\sin.\frac{1}{2}t\tan.\beta}};$$

$$(2) \text{ and II.} \qquad \tan.\gamma = \sqrt{n} = \frac{\cos.\alpha}{\sqrt{2.\sin.\frac{1}{2}t\cos.\beta}};$$

(*b*) n positive, and greater than 1.
Let $n = \sec.^2\gamma$
$n-1 = \tan.^2\gamma$.

$$(3) \text{ and II.} \qquad \sec.\gamma = \sqrt{n}\ \frac{\tan.\alpha}{\sqrt{2.\sin.\frac{1}{2}t\tan.\beta}}.$$

(*c*) n positive, and less than 1.
Then $(1-n)$ is to be investigated.
Let $n = \cos.^2\gamma$
$(1-n) = \sin.^2\gamma$

$$(3) \text{ and II.} \qquad \cos.\gamma = \sqrt{n} = \frac{\tan.\alpha}{\sqrt{2.\sin.\frac{1}{2}t\tan.\beta}}.$$

Now, $t - c = \dfrac{c-r}{n-1}$, or $t = c + \dfrac{c-r}{n-1}$;

Hence,

$$(a) \quad t = c + \frac{c-r}{-\sec.^2\gamma};$$

$$(b) \quad t = c + \frac{c-r}{\tan.^2\gamma};$$

$$(c) \quad t = c + \frac{c-r}{-\sin.^2\gamma}.$$

The application of Logarithms to the above values of α, β, γ, &c. readily furnishes the explanation of the quantities P, L, Q, &c. and the signs of these quantities enable us to obtain the final conclusion exact.

When the Observations are on different sides of Noon, by a similar process we obtain $n = \dfrac{1-\tan.D\cot.l}{\text{vsin}.t+\sin.t\tan.m}$

Here m is negative with respect to its former value, and with respect to t,

$\text{vsin}.t + \sin.t\tan.m = 2\sin.^2\frac{1}{2}t\,(\cot.\frac{1}{2}t\tan.m + 1)$.
Let $\cot.\frac{1}{2}t\tan.m = \tan.^2\beta$;
Then $\text{vsin}.t + \sin.t\tan.m = 2\sin.^2\frac{1}{2}t\sec.^2\beta$.
The rest as before.